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NP-Completeness of Graph Decomposition Problems

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The H-decomposition problem for a fixed graph H is stated as follows: Can an input graph G be represented as an edge disjoint union of subgraphs, all of which are isomorphic to H? Although H-decomposition problems have been the subject of extensive mathematical research for many decades, even the complexity status of such problems is yet unknown, except for a few families of graphs. H. I. Holyer conjectured that H-decomposition is NP-complete whenever H is connected and has at least 3 edges. The above was proved, however, only for a limited class of graphs H: complete graphs, simple paths, and simple circuits. Holyer's conjecture is proved here for a large family of graphs which contains all trees.

1. INTRODUCTION

Let G and H be two graphs. An H-decomposition of G is a representation of G as an edge disjoint union of subgraphs, all of which are isomorphic to H. Graph decomposition was, and still is, a quite popular research area. Intensive research has been done on many special cases, beginning with the classical work on "Steiner triple systems" (H is a triangle and G a complete graph) (Steiner, 1853) and proceeding with hundreds of papers up to the present (see Bermond and Sotteau (1975) for a partial list of references).

One of the most important results in the area is the following, obtained by R. M. Wilson (1976): For every graph $H = (V, E)$ and an integer n, $n \geq n_0(H)$, the complete graph K_n has an H-decomposition if and only if $|E|$ divides $\binom{n}{2}$ and $n - 1$ is divisble by g.c.d. $\{d(x) \mid x \in V\}$. Wilson's theorem implies that the existence of H-decomposition of the complete graph K_n is decidable for any fixed graph H, in $O(\log(n))$ time. Practically, even for the simple case where H is the complete graph on 6 vertices, there are still

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several dozens of integers n (all in the interval 100–1000) for which H-decomposability of K_n is yet undetermined. (The case where H is a small complete graph and $G = K_n$ was the first and most intensively studied.) Wilson's theorem points out a family of graph decomposition problems, which can be solved in polynomial time. Not many other results of that character are known. Recently Y. Caro (1985) presented a polynomial time algorithm to solve decomposition problems where both G and H are trees.

We are interested here in the class of problems, each determined by a fixed graph H, where the other graph G is given as input.

DEFINITION. For a fixed graph H, the H-decomposition problem is stated as follows: Can an input graph G be represented as an edge disjoint union of subgraphs, all of which are isomorphic to H.

I. Holyer (1981) proved that H-decomposition is NPC for every complete graph $H = K_n$, $n \geq 3$. He also conjectued NP-completeness whenever H consists of at least 3 edges. In its general form the assertion of that conjecture is false (assuming $P \neq NP$). Even before Holyer stated his conjecture, Brouwer and Wilson (1980), in an unpublished paper, gave a polynomial time algorithm (although their result is not stated in that form) for the case where $H = tK_2$ (the union of t disjoint edges). The same result was obtained later, independently, by N. Alon (1983), after the case $H = 3K_2$ had been studied by Bialostocki and Roditty (1982). Recently, Favaron *et al.* (1985) have obtained a similar result for the case where H is the disjoint union of a single edge and a simple path with 2 edges. Holyer's conjecture might still hold if restricted to the case where H is connected, or equivalently, contains a connected component with at least 3 edges (for the equivalence see Section 3 of this paper).

While discussing known results, mention should be made of a closely related topic, the H-factorization problem, where vertices, rather than edges, play the main role: Given an input graph $G = (V, E)$, is there a collection of vertex disjoint subgraphs of G, isomorphic to a fixed graph H, such that the union of their vertex sets is V? H-factorization is also known as generalized matching. A complete matching in a graph G is in fact a K_2-factorization. The complexity status of this class of problems was studied by Kirkpatrick and Hell (1978), who showed that H-factorization is NPC, whenever H contains a connected component of at least three vertices and it is polynomial otherwise.

In Section 2 of this paper we prove Holyer's conjecture for a family of connected graphs which contains all trees. Our theorem generalizes previous partial results regarding some specific trees. D. Leven (unpublished results) proved that S_n-decomposition, $n \geq 3$ (see 2.1 for the definition) is NPC and I. Holyer (1980) obtained a similar result for simple paths. Our proof was first designed for the case where H is a tree, and hence it is highly dependent on the existence of vertices of degree 1. Later we observed that it can also be applied to all graphs which contain a small "starlike" substructure. The proof still calls for vertices of degree 1 and thus we cannot use it to cope with the general case where H does not have such a vertex.

In Section 3 we show that H-decomposition is NPC if there is a connected component H' of H for which H'-decomposition is NPC.

2. The Main Result

THEOREM 1. *If T is a connected graph with more than 2 edges, in which there exists a vertex r such that all the vertices adjacent to r, except at most one of them, are of degree one, then T-decomposition is NPC.*

Before stating the proof note that Theorem 1 covers the case where T is a tree, letting r be the second vertex along a simple path of maximal length.

For any graph T the problem is obviously in NP, thus it suffices to show NP-hardness. The proof varies as different families of graphs T are considered. The main scheme of the proof is presented while dealing with the simple case where T is a star.

Proof of Theorem 1.

2.1. $T = S_n$, $n \geq 3$. S_n, the star of order S_n, is the complete bipartite graph $K_{1,n}$. The vertex of degree n is called the *center* of the star. Several proofs are known for this case, some of which are simpler than the one presented here. The following proof however, provides a general scheme which is later applied to more general graphs.

The known NPC problem for which we show polynomial reduction into S_n-decomposition is the "exact hitting set for 3-subsets" (3-EHS, for short), defined as follows:

Given a finite set U and a collection $\mathbb{A}$ of 3-element subsets of U, is there a set $X \subseteq U$ such that $|X \cap A| = 1$ for every $A \in \mathbb{A}$?

The problem is also known as "one in three 3sat without negated literals." See Garey and Johnson (1979), p. 259, Lo4.

Let $I = (U, \mathbb{A})$ be an instance of 3-EHS. In polynomial time, a graph $G_{S_n}(I)$ can be constructed for which S_n-decomposition is equivalent to 3-EHS on I. The following is a set of instructions for constructing $G_{S_n}(I)$:

Let every 3-tuple $A \in \mathbb{A}$ be represented by two disjoint stars: a copy of S_{n-1} and a copy of S_{n-2}. The centers of these stars are denoted by A^+ and

A^-, respectively. Let $k(x)$ denote the number of 3-tuples $A \in \mathbb{A}$ which contain the element x of U. For every $x \in U$ construct an $(n-1)$-regular, connected, bipartite graph $G_x(S_n)$ on two independent sets S_x^+ and S_x^-, each consisting of $k' = \max\{k(x), n\}$ vertices. Label the vertices of S_x^+ as $u_1, \ldots, u_{k'}$ and those of S_x^- as $v_1, \ldots, v_{k'}$. In the case where $n > k$ choose for every $i > k$ u_i *and* v_i to be nonadjacent and add the edge (u_i, v_i) to the graph. Figure 1a presents two examples for such graphs, one with the parameters $k = 3$, $n = 4$ and another with $k = 3$, $n = 3$. (The roles of the extra edges and the $A_k(S_n)$ notation will be clarified later.) For every pair x, A, where $x \in A$, choose a distinct pair of vertices (u_i, v_i) from $G_x(S_n)$ and add one edge e_{x,A^+} with end vertices (A^+, v_i) and another—e_{x,A^-}—with end vertices (A^-, u_i) (See Fig. 1b). The obtained graph is $G_{S_n}(I)$.

Let us now verify that S_n-decomposition of $G_{S_n}(I)$ is equivalent to 3-EHS on I: The degree in $G_{S_n}(I)$ of every vertex of $S_x^+ \cup S_x^-$ is n. For $A \in \mathbb{A}$ the degree of A^+ is $n + 2$ and that of A^- is $n + 1$. Assume $X \subseteq U$ is a solution of I. That is, for every A $\in \mathbb{A}$ exactly one element of A belongs to X and two elements do not. For every $x \in X$ remove the copies of S_n centered at every $u \in S_x^-$. For every $y \notin X$ remove the copies of S_n centered at every $v \in S_y^+$. Take $A \in \mathbb{A}$, since $|A \cap X| = 1$; the removed stars contain exactly one edge e_{x,A^-} for a certain $x \in X$, incident to A^- and two edges e_{y,A^+} and e_{z,A^+}, $y, z \notin X$, incident to A^+. Thus, the remaining edges form a copy of S_n centered at every vertex A^+ and a copy of S_n centered at every A^-. Together with the removed stars an S_n-decomposition of $G_{S_n}(I)$ is completed.

On the other hand, assume that an S_n-decomposition of $G_{S_n}(I)$ exists. Take an edge e of one of the bipartite subgraphs $G_x(S_n)$. Since e is covered by the decomposition, at least one of its end vertices is the center of a star in the decomposition (we call such a vertex a center of the decomposition or simply a center). The degree (in $G_{S_n}(I)$) of every vertex of $G_x(S_n)$ is exactly n, hence only one of the end vertices of e is a center of the decomposition. The set of centers is thus independent in $G_x(S_n)$ and covers all its edges. $G_x(S_n)$ is connected so there are exactly two possibilities: Either every $v \in S_x^+$ is a center and no $u \in S_x^-$ are such, or every $u \in S_x^-$ is a center and no $v \in S_x^+$ are such. Define $X = \{x \in U \mid u \in S_x^- \Rightarrow u \text{ is a center}\}$. Focus on a 3-tuple $A \in \mathbb{A}$; the vertex A^- of degree $n + 1$ is necessarily the center of one decomposition star. The $n + 1^{\text{st}}$ edge indicident to A^- belongs to a star centered at a vertex $u \in S_x^-$ for some $x \in X$. Thus A contains exactly one element $x \in X$ and hence X is indeed a solution of I. ∎

Take any element $x \in U$ with $k(x) = k$. The subgraph of $G_x(S_n)$ consisting of all edges with at least one end vertex in $G_x(S_n)$ is determined, up to graph isomorphism, by the parameter k. Let us denote such a graph by

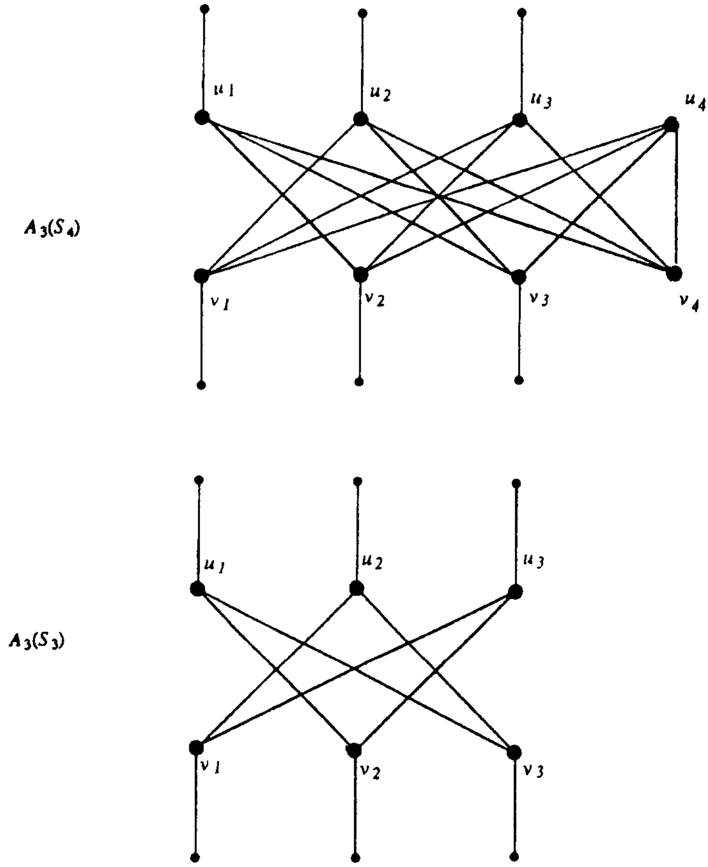


FIGURE 1a

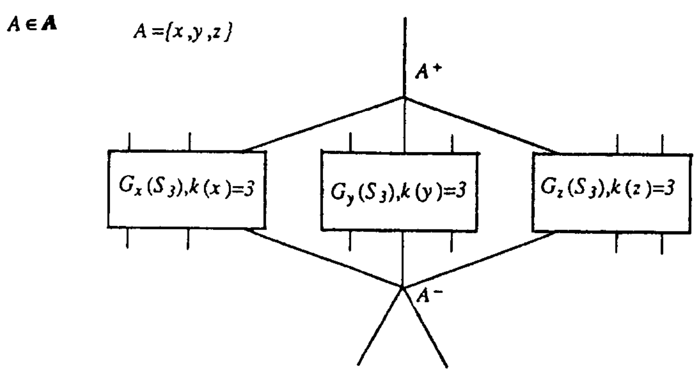


FIGURE 1b

$A_k(S_n)$ (see Fig. 1a). This last definition will be used in later subsections.

To handle more complex graphs some further notation and definitions are required:

A leaf: A vertex of degree 1.

A final edge: An edge incident to a leaf. The final edge incident to a leaf v is denoted by f_v.

T-DE: Let $G = (V, E)$, $S \subseteq V$. We say that $\bar{G} = (\bar{V}, \bar{E})$ is a *T-decomposable extension* (abbreviated as *T-DE*) of G on S, if the following holds:

1. $V \subseteq \bar{V}$, $E \subseteq \bar{E}$ and the removal of S disconnects G from the rest of $\bar{G}$; that is, $\bar{G}$ is obtained by attaching new components to G on vertices of S.
2. There exists a T-decomposition of $\bar{G}$.
3. Every copy of T in any T-decomposition of $\bar{G}$ contains at least one edge from E.

T-RE: Let $\bar{G}$ be a *T-DE* of G on S. Out of a T-decomposition of $\bar{G}$, remove the copies of T which are entirely contained in G. The set of edges of G not covered by these removed subgraphs is called a *T-remainder* (*T-RE*) of G on S.

When using the last two definitions in the proof we strongly rely on the following simple observations:

PROPOSITION 1. *If G is a subgraph of $\bar{\bar{G}}$, S is a set of vertices disconnecting G from the rest of $\bar{\bar{G}}$, and $\bar{\bar{G}}$ has a T-decomposition then $\bar{\bar{G}}$ contains a subgraph $\bar{G}$, which is a T-DE of G on S, and if the edges of $\bar{G}$ are removed from $\bar{\bar{G}}$ then the remaining subgraph has a T-decomposition.*

Proof. Out of a T-decomposition of $\bar{\bar{G}}$ take the copies of T which contain edges of G. If the union of those copies is not a *T-DE* of G then it has another T-decomposition in which some copies of T are edge disjoint from G. Remove these copies and proceed in that fashion, until a *T-DE* is reached.

An immediate consequence of Proposition 1 is:

PROPOSITION 2. *If G is a subgraph of $\bar{\bar{G}}$, S is a set of vertices disconnecting G from the rest of $\bar{\bar{G}}$, and $\bar{\bar{G}}$ has a T-decomposition then for some T-RE G' of G on S,* $(\bar{\bar{G}} - G) \cup G'$ still has a T-composition. That is, replacing G by its *T-RE* G' preserves the exstience of a T-decomposition.

Let us now analyze the proof for S_n to learn the exact role of the various subgraphs of $G_{S_n}(I)$. The relevant property of the stars S_{n-1} and S_{n-2}, centered at A^+ and A^- is the following simple fact: The only S_n-*DE* of S_{n-1}, respectively S_{n-2}, on its center is obtained by attaching an additional final edge, respectively two final edges, incidenct to the center. The only property of the subgraphs $A_k(S_n)$ which is relevant to the proof is: There are exactly two S_n-RE's of $A_k(S_n)$ on $\{A^+, A^- \mid x \in A\}$, each of which contains $k = k(x)$ final edges, namely $\{e_{x,A^+} \mid x \in A\}$ and $\{e_{x,A^-} \mid x \in A\}$. It is now quite obvious how the argument of the proof for stars yields the following lemma:

LEMMA 2.1. *T-decomposition is NPC if there exist graphs T', T'' and a graph $A_k(T)$, constructed in polynomial time (in k) for every positive integer k, such that the following hold:*

1. *T' and T'' each has a vertex, called its center, such that the only T-DE of T', respectively of T'', on the center is obtained by attaching an additional final edge, respectively two additional final edges, incident to the center.*

2. *There are two disjoint sets P and Q, each consisting of k leaves of $A_k(T)$, such that the only T-RE's of $A_k(T)$ on $P \cup Q$ are $F_P = \{f_p \mid p \in P\}$ and $F_Q = \{f_q \mid q \in Q\}$.*

Let T be a connected graph, which contains a vertex r, satisfying the conditions of Theorem 1. Define $n = n(G)$ by $n = d(r) - 1$. Let the nonleaf vertex adjacent to r be denoted by h and let H stand for the subgraph of T, obtained by the removal of the edges incident to r. Since the case where T is a star has already been accounted for, we also assume that H is not empty.

The existence of T', T'' as required for Lemma 2.1 is straightforward: T' is the subgraph of T obtained by the removal of a final edge incident to r. The vertex r is the center of T'. To obtain T'' take two copies of T' which are disjoint, except for having a common center which is also the center of T'' (see Fig. 2).

To verify that T', as it has just been defined, satisfies the condition of

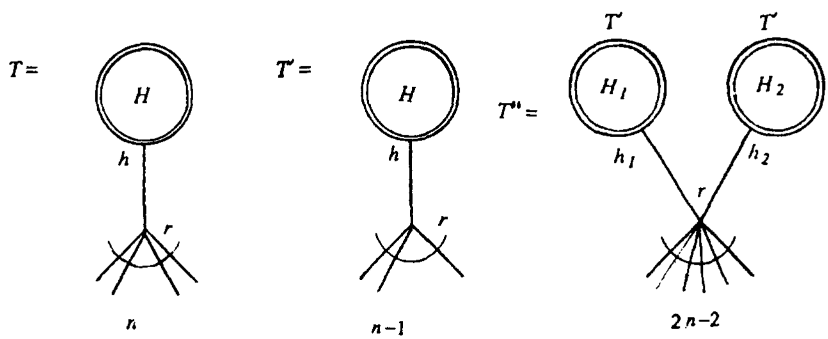


FIGURE 2

Lemma 2.1, take a T-decomposition of a T-DE of T' on its center. The nonempty subgraph H is connected to the center through a single edge, hence its edges are all contained in a single copy of T in the decomposition. The only way to complete H into a copy of T through the vertex r is by attaching a new final edge to the center, as required. The analogous property of T'' is verified similarly.

The construction of $A_k(T)$ is still required to complete the proof of Theorem 1. For this purpose we first present a new building block, the graph $C(T)$. This graph is obtained by attaching an additional final edge incidenct to the vertex r of T and labelling one of the leaves adjacent to r by u and another one by v. Let u and v be called the *labeled* vertices of $C(T)$. $C(T)$ is used for the following property: There are exactly two T-RE's of $C(T)$ on $\{u, v\}$ namely $\{f_u\}$ and $\{f_v\}$. The validity of the last statement is derived by the same argument we used to verify the analogous property of T'.

We now present two alternatives to complete the construction of $A_k(T)$, each of which is used for different values of the parameter n.

2.2. $n \geq 3$. Construct the graph $B(T)$ as follows (see Fig. 3): Take disjoint copies of H and T'. Make a contraction of the vertex h of H and the center of T' into a single vertex c and attach two additional final edges incident to c: f_l and f_m. The leaves l and m are called the *legs* of $B(T)$. Denote by $\bar{B}(T)$ the graph obtained by attaching to $B(T)$ two copies of S_n: S^1 and S^2 centered at l and m, respectively. Our use of $B(T)$ relies on the following: The only subgraphs of $\bar{B}(T)$ which are T-DE's of $B(T)$ on its legs are $B(T) \cup S^1$ and $B(T) \cup S^2$. To verify the last statement, T' is completed to a copy of T by an edge incident to c. If this edge is either f_l or f_m than a T-DE of the remaining subgraph takes an S_n centered at the other leg. If the edge is taken from H then the number of available edges is too small for two additional copies of T and the only two ways to complete the remaining subgraph to one copy of T is by taking either S^1 or S^2.

$A_k(T)$ is constructed as follows: Take the graph $A_k(S_n)$, defined in Section 2.1. Replace every edge (a, b) of this graph by a copy of $C(T)$ whose labeled vertices are a and b. On every pair (u_i, v_i), $1 \leq i \leq \max\{k, n\}$,

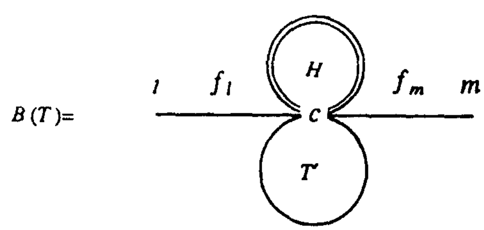


FIGURE 3

construct a copy of $B(T)$ for which u_i and v_i are the legs. Let $P = \{A^+ \mid x \in A\}$ and $Q = \{A^- \mid x \in A\}$.

To see that a *T-RE* of $A_k(T)$ on $P \cup Q$ is either F_P or F_Q we should follow the discussion in the last paragraph of the proof for stars in Section 2.1. Note that after each copy of $C(T)$ is repleaced, according to Proposition 2, by its *T-RE*, which is a single edge, it can be considered as the original edge of $A_k(S_n)$, which is now "disconnected" from one of its original end vertices. This forces the *T–DE*'s of the $B(T)$'s to be obtained by taking a copy of S_n centered at either u_i or v_i for every pair (u_i, v_i) as was the case in the proof for stars.

2.3. $n \leq 2$. In order to use a similar scheme these graphs require a replacement of $G_x(S_n)$ since there is no 1-regular connected graph large enough. The proof's scheme is presented as a lemma.

LEMMA 2.2. *T-decomposition is NPC if T is as required in Theorem* 1 *and there exists a graph $F(T)$ which satisfies the following*:

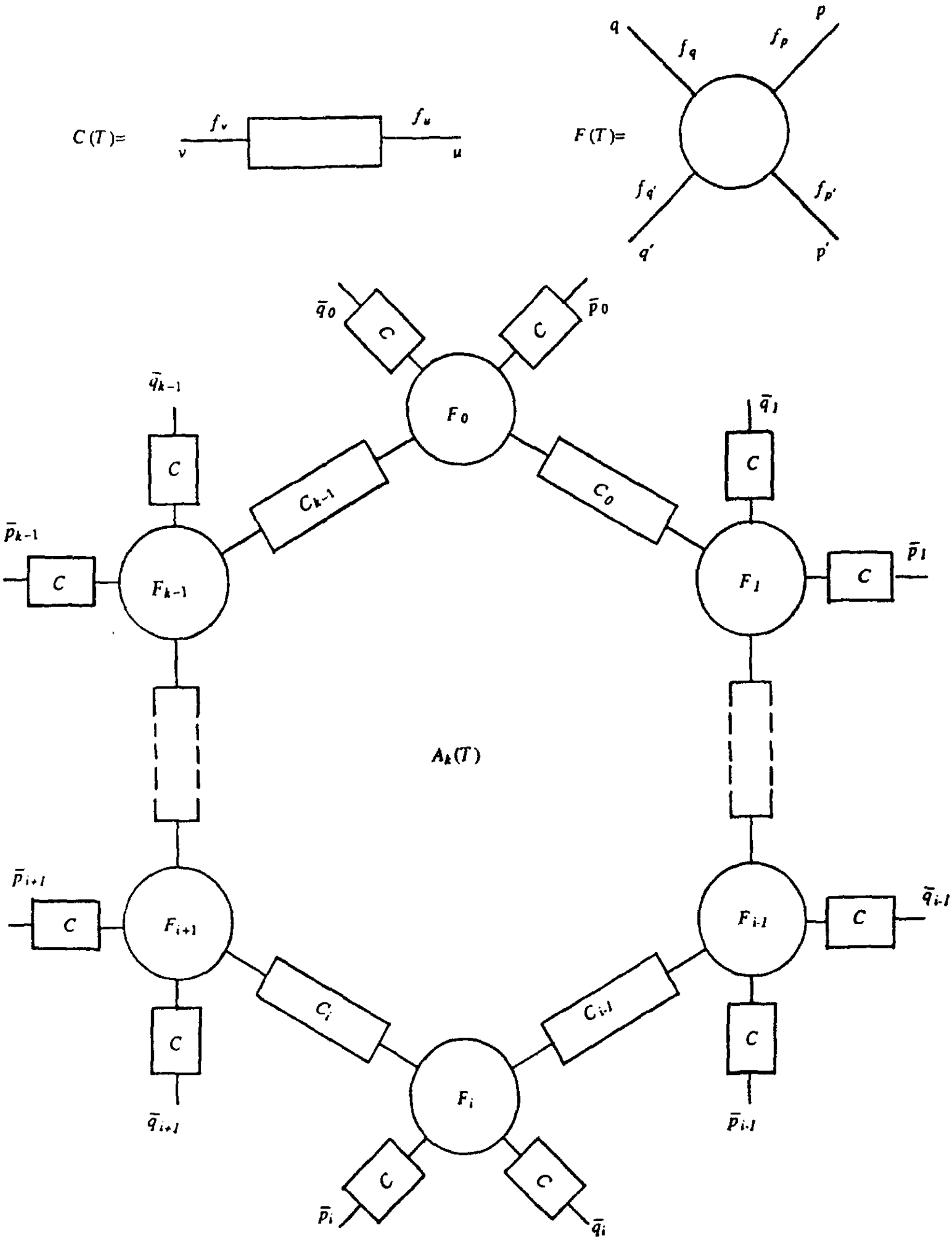
There are two pairs of leaves of $F(T)$, (p, p') and (q, q'), such that the only two T-DE's of $F(T)$-$\{f_p, f_{p'}, f_q, f_{q'}\}$ which are subsets of $F(T)$ are $F(T)$-$\{f_p, f_{p'}\}$ and $F(T)$-$\{f_q, f_{q'}\}$. (The set S on which the *T-DE*'s are constructed is defined by the attached edges.)

Proof of the lemma. Take a sequence of k disjoint copies of $C(T)$ denoted $C_0, C_1, \ldots, C_{k-1}$ and k copies of $F(T)$: $F_0, \ldots, F_{k-1}$. Label the copies of p, p', q, q' in F_i and u, v in C_i as $p_i, p_i', q_i, q_i', u_i$, and v_i, respectively. For every i, $0 \leq i \leq k-1$, replace the edges $f_{p_i'}$ of F_i and f_{u_i} of C_i by a single edge, joining the two inner end vertieces of the replaced edges. Similarly, identify f_{v_i} and $f_{q_{i+1}'}$. Take the index i modulo k so C_{k-1} is connected back to F_0. Now on each final edge f_{p_i} attach an additional copy of $C(T)$, again by means of identification of this edge with the copy of f_u. The copy of the vertex v in the attached graph is denoted $\bar{p}_i$. In a similar way, replace each final edge f_{q_i} by a copy of $C(T)$ and label the corresponding copy of the vertex v as $\bar{q}_i$. The obtained graph is $A_k(T)$ (see Fig. 4). Denote $P = \{\bar{p}_i \mid 0 \leq i \leq k-1\}$, $Q = \{\bar{q}_i \mid 0 \leq i \leq k-1\}$ and $F_P = \{f_{\bar{p}} \mid \bar{p} \in P\}$, $F_Q = \{f_{\bar{q}} \mid \bar{q} \in Q\}$.

To complete the proof of the lemma we show that $A_k(T)$ satisfies Lemma 2.1. Take a *T-DE* of $A_k(T)$, after each $C(T)$ is replaced according to Proposition 2 by a single edge and each copy of $F(T)$ (or what is left of it) is disconnected from the rest of the graph. If a *T-RE* of one of the C_i's is the edge through which it is connected to F_i, respectively to $F_i + 1$, then for every C_j its *T-RE* is the edge which connects it to F_j, respectively to F_{j+1}. The required result immediately follows. ■

We now explicitly construct $F(T)$, dealing separately with two subcases:

2.3.1. $n = 2$. Take disjoint copies of T, T', and S_3 (we refer to these


C(T)=
v
f_v
f_u
u
F(T)=
q
f_q
p
f_p
f_q'
q'
f_p'
p'
A_k(T)
F_0
F_1
F_k-1
F_i-1
F_i+1
F_i
C_0
C_k-1
C_i
C_i-1
C
q̄_0
p̄_0
q̄_k-1
p̄_k-1
q̄_1
p̄_1
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p̄_i-1
p̄_i
q̄_i

FIGURE 4

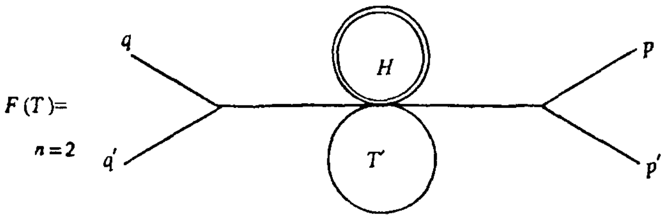

FIGURE 5

copies as T, T', and S_3). Label the two leaves adjacent to r in T as p and p'. Label the leaves of S_3 as q, q', and v. Construct a connected graph $F(T)$ by a contraction of the vertices h of T, v of S_3 and the center of T' into a single vertex c. (See Fig. 5).

To verify that $F(T)$ satisfies the conditions of Lemma 2.2 note that a *T-DE* of T' requires an additional edge incident to c. If this edge is not taken from the subgraph H of T our claim is obvious. If it is taken from H, then the only subgraph isomorphic to T which can be embedded within the remaining edges is either $F(T)$-$\{f_p, f_{p'}\}$ or $F(T)$-$\{f_q, f_{q'}\}$. ∎

2.3.2. $n = 1$. (See Fig. 6.) Take two copies of T: T_1 and T_2. Make a contraction of the two copies of the vertex r, r_1 in T_1 and r_2 in T_2, into a single vertex. Label the leaves adjacent to the contracted vertex as p and p'. Do the same with an additional pair of isomorphic copies of T, T_3 and T_4; this time label the corresponding leaves as q and q'. Now make a contraction of the two copies of the subgraph H from T_2 and T_3 into a single copy of H by identifying each pair of corresponding vertices and each pair of corresponding edges. The obtained graph is $F(T)$.

A *T-RE* of $F(T)$ on $\{p, p', q, q'\}$ is easily verified to be either $\{f_p, f_{p'}\}$ or $\{f_q, f_{q'}\}$ or another path of length 2 which is not a subset of $\{f_p, f_{p'}, f_q, f_{q'}\}$. This directly implies the required property of $F(T)$.

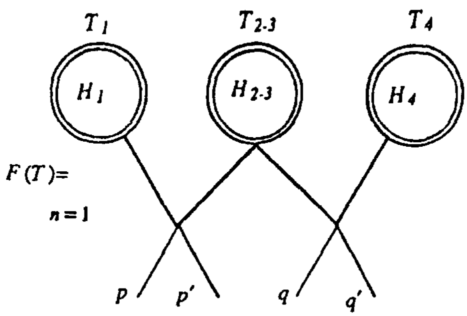

FIGURE 6

3. H-DECOMPOSITION FOR DISCONNECTED GRAPHS

The following extends our results to a large family of disconnected graphs.

THEOREM 2. *Let H be the disjoint union of the connected graphs H_1, $H_2, \ldots, H_k$. If H_1-decomposition is NPC, then H-decomposition is NPC.*

Proof. Suppose we have a polynomial time algorithm for the H-decomposition problem. Given $G = (V, E)$, it can be used to determine whether H_1 decomposes G. Check if $e(H_1)$ (the number of edges in H_1) divides $e(G)$, and if so construct L from G by adding $e(G)/e(H_1)$ disjoint copies of $H_2, \ldots, H_k$ to G. We claim that H_1 dcomposes G if and only if H decomposes L. Indeed, if H_1 decomposes G, clearly H decomposes L. Suppose H decomposes L.

Case 1. There is some H_i, such that H_1 does not decompose H_i. Hence there is such an H_i with a minimum number of edges, so no other H_j decomposes it and thus in the H-decomposition of L all the copies of H_i cover themselves. Now we can delete H_i from both H and L and apply induction on k.

Case 2. H_1 decomposes each H_i. Hence in the H-decomposition of L, G is decomposed into H_i's and this gives a decomposition of G into copies of H_1. ∎

Even if Holyer's conjecture is true for connected graphs, it is not yet clear whether the last theorem can be extended to an "if and only if" theorem. We believe, however, that H-decomposition is polynomial if H has no connected component with more than 2 edges. As already mentioned in the Introduction, there are some partial results which support this.

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